Chapter 12

Problem 1. *(a) Find the plane going through the points* $A = (1, 0, 0)$*,* $B = (2, 0, -1)$ *and* $C = (1, 4, 3)$*.*

(b) Find the area of triangle ΔABC*.*

Solution

1. Two displacement vectors in the plane are $\langle 1, 0, -1\rangle$ and $\langle 0, 4, 3\rangle$ (from $B - A$ and $C - A$ respectively). Hence the normal vector to the plane is the cross product of these two vectors, which is $\vec{n} = \langle 4, -3, 4\rangle$. The equation of the plane is then $\vec{n} \cdot \langle x, y, z\rangle = \vec{n} \cdot P$ for any point in the plane, e.g. A. Thus,

$$4x - 3y + 4z = 4$$

   works.

2. The area is $\frac{1}{2}|\vec{n}| = \frac{1}{2}\sqrt{4^2 + 3^2 + 4^2} = \frac{1}{2}\sqrt{41}$.

Problem 2. *Are the lines given by the symmetric equations*

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

and

$$\frac{x-1}{6} = \frac{y-3}{-1} = \frac{z+5}{2}$$

parallel, skew, or intersecting?

Solution The first line has direction $\langle 2, 3, 4\rangle$ and the second has direction $\langle 6, -1, 2\rangle$. These are obviously not scalar multiples of each other, so the lines are not parallel. It's also easy to check that there is no triple (x, y, z) that satisfies both equations simultaneously, so they are nonintersecting. That is, skew.

Problem 3. *Find the distance between the planes* $3x + y - 4z = 2$ *and* $3x + y - 4z = 26$*.*

Solution Take any displacement between a point on the first plane, such as $(0, 2, 0)$, and a plane in the second, such as $(0, 26, 0)$ and compute its (scalar) component of the point along the normal vector $\langle 3, 1, -4\rangle$. So we have (here $\vec{b} = (0, 24, 0)$)

$$|\operatorname{comp}_{\vec{n}} \vec{b}| = \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|} = \frac{24}{\sqrt{26}}$$

Problem 4. *Identify and sketch the surfaces:*

(a) $x = y^2 + z^2 - 2y - 4z + 5$

(b) $y = z^2$

Solution

(a) Rearranging, this is

$$x = (y-1)^2 + (z-2)^2$$

which is an elliptic paraboloid based at $(0, 1, 2)$ and whose axis points in the direction of the x-axis.

(b) This is a cylinder, which consists of a copy of the parabola $y = z^2$ copied once for each value of x. That is, if you look in the direction of the x-axis, you see only a parabola. (Sketch omitted).

Chapter 13

Problem 5. *Consider* $\vec{r}(t) = t\vec{i} + \cos \pi t \vec{j} + \sin \pi t \vec{k}$*.*

(a) Sketch the graph of $\vec{r}$*.*

(b) Find the tangent line at $t = 0$*.*

(c) Find the integral $\int_0^1 \vec{r}(t)dt$.

(d) Find the arclength of $\vec{r}$, $0 \le t \le 1$.

Solution

(a) This is a helix of radius 3, growing in the direction of the x-axis (sketch omitted).

(b) The direction vector is $\vec{r}'(0)$; since $\vec{r}'(t) = \vec{i} - \pi \sin \pi t \vec{j} + \pi \cos \pi t \vec{k}$, this will reduce to $\vec{i} + \pi \vec{k}$. It passes through the point $(0, 1, 0)$, so

$$L(t) = \langle 0, 1, 0\rangle + t\langle 1, 0, \pi\rangle$$

is one such line.

(c) Integrate componentwise:

$$\begin{aligned}\int_0^1 \langle t, \cos \pi t, \sin \pi t\rangle dt &= \langle \int_0^1 tdt, \int_0^1 \cos \pi t dt, \int_0^1 \sin \pi t dt\rangle \\ &= \langle \frac{1}{2}, 0, \frac{2}{\pi}\rangle\end{aligned}$$

(d) Integrate $|\vec{r}'|$: $|\vec{r}'(t)| = \sqrt{1+\pi^2}$. Integrating this on $[0, 1]$ gives $\sqrt{1+\pi^2}$.

Problem 6. *A ball is thrown at an angle of $60°$ above the horizon at a speed of 100 m/s. Where does it hit the ground?*

Solution Let $\vec{r}$ denote the position; we have $\vec{r}'' = -gj$, with g being the gravitational constant, roughly 10 m/s^2. The initial velocity is the vector $100\langle \cos 60, \sin 60\rangle$, so

$$\vec{r}'(t) = \langle 100 \cos 60, 100 \sin 60 - gt\rangle$$

Set the initial position to be the origin and integrate again:

$$\vec{r}(t) = \langle 100 \cos 60t, 100 \sin 60t - \frac{1}{2}gt^2\rangle$$

Since $\sin 60 = \sqrt{3}/2$ and $g = 10$, we can write this as

$$\vec{r}(t) = \langle 50t, 50\sqrt{3}t - 5t^2\rangle$$

The y-value is zero when it hits the ground, and solving gives $t = 0$ or $t = 10\sqrt{3}$. Putting this in the x component gives that the ball hits the ground at a distance of $500\sqrt{3}$ units downrange.

Chapter 14

Problem 7. *Find and sketch the domain of $f(x, y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 - x^2}$.*

Solution The region is described by the inequalities $1 - x^2 \ge 0$ and $4 - x^2 - y^2 \ge 0$. That is, this is the region bounded by the circle $x^2 + y^2 = 4$ and contained within the strip $-1 \le x \le 1$.

Problem 8. *Evaluate the limits, or show why they do not exist:*

(a)

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2}$$

(b)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y e^y}{x^4 + 4y^2}$$

Solution Neither limit exists.

(a) Approaching along the coordinate axes suggests a limit of 0, while approaching along the line $y = x$ gives

$$\lim_{x \to 0} \frac{2x^2}{x^2 + x^2} = 1 \neq 0$$

(b) Approaching along the coordinate axes suggests a limit of 0, as does approaching on any straight line. But along the parabola $y = x^2$,

$$\lim_{x \to 0} \frac{x^4 e^{x^2}}{x^4 + 4x^4} = \frac{1}{5} \neq 0$$

so the limit does not exist.

Problem 9. *Let* $\cos(xyz) = 1 + x^2y^2 + z^6$. *Find* $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$.

Solution Differentiating in x first, we have

$$-\sin(xyz)xy\frac{\partial z}{\partial x} = 2xy^2 + 6z^5\frac{\partial z}{\partial x}$$

This can be easily solved for $\partial z/\partial x$, giving

$$\frac{\partial z}{\partial x} = \frac{-2xy^2}{6z^5 + xy\sin(xyz)}$$

Problem 10. *Let* $f(x, y) = x^2e^{-y}$.

(a) Find the directional derivative of f *at the point* $(-2, 0)$ *in the direction of the origin.*

(b) In which direction is the maximum rate of change?

Solution

(a) The gradient of f is $\langle 2xe^{-y}, -2xe^{-y}\rangle$, and the value at $(-2, 0)$ is $\langle -4, -4\rangle$. The direction vector towards this origin, normalized to be a unit vector, is $\vec{u} = \langle 1, 0\rangle$; the directional derivative is then

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = -4$$

(b) The maximum rate of change occurs in the direction of the gradient, which is $\langle -1/\sqrt{2}, -1/\sqrt{2}\rangle$.

Problem 11. *Let* $N = (p + q)/(p + r)$ *and* $p = u + vw$, $q = v + uw$, $r = w + uv$. *Find* $\frac{\partial N}{\partial u}$.

Solution

$$\begin{aligned}\frac{\partial N}{\partial u} &= \frac{\partial N}{\partial p}\frac{\partial p}{\partial u} + \frac{\partial N}{\partial q}\frac{\partial q}{\partial u} + \frac{\partial N}{\partial r}\frac{\partial r}{\partial u} \\ &= \frac{r - q}{(p + r)^2} \cdot 1 + \frac{1}{p + r} \cdot w + \frac{-(p + q)}{(p + r)^2} \cdot v\end{aligned}$$

Replacing p, q, r with their expressions in terms of u, v, w finishes the problem.

Problem 12. *Find the absolute maximum and minimum values of* $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$ *on the disk* $x^2 + y^2 \leq 4$.

Solution On the boundary, the function is equal to $e^{-4}(4 + y^2)$; since the latter part is a paraboloid, this is clearly minimized at $y = 0$ (with value $4e^{-4}$) and maximized at $y = \pm 2$ (with value $8e^{-4}$).

For the interior, we use the critical points:

$$\begin{aligned}0 = f_x &= e^{-x^2-y^2}(2x) + (x^2 + 2y^2)e^{-x^2-y^2}(-2x) \\ 0 = f_y &= e^{-x^2-y^2}(4y) + (x^2 + 2y^2)e^{-x^2-y^2}(-2y)\end{aligned}$$

Simplifying leads to

$$\begin{aligned}2x(1 - x^2 - 2y^2) &= 0 \\ 2y(2 - x^2 - 2y^2) &= 0\end{aligned}$$

We now have a few cases:

- If $x = 0$ and $y = 0$, the value is 0.
- If $x = 0$ and $y \neq 0$, then $2 - 2y^2 = 0 \implies y = \pm 1$. Either way, the value of f is $f(0, \pm 1) = 2e^{-1}$.
- If $y = 0$ and $x \neq 0$, then $1 - x^2 = 0 \implies x = \pm 1$. Either way, the value of f is $f(\pm 1, 0) = e^{-1}$.
- If x and y are both non-zero, we have $1 - x^2 - 2y^2 = 0$ and $2 - x^2 - 2y^2 = 0$, which obviously has no solution.

Ordering these values by size, the minimum is 0 attained at the origin and the maximum is $2e^{-1}$ attained at $(0, \pm 1)$.

Chapter 15

Problem 13. *Evaluate*

$$\int_0^1 \int_x^1 e^{y^2} \, dy \, dx$$

Solution The region described is a triangle with vertices $(0, 0)$, $(1, 1)$ and $(0, 1)$. The inequalities $0 \leq x \leq 1, 0x \leq y \leq 1$ can be rewritten as $0 \leq y \leq 1$, $0 \leq x \leq y$, so changing the order of integration yields

$$\int_0^1 \int_0^y e^{y^2} \, dx \, dy = \int_0^1 y e^{y^2} \, dy = \frac{1}{2} e^{y^2} \Big|_0^1 = \frac{e-1}{2}$$

Problem 14. *Find the volume of the region bounded by $x^2 + y^2 = 4, z = 0$ and $y + z = 3$.*

Solution This is a cylindrical region lying between two planes, so we use cylindrical coordinates. Here, $0 \leq r \leq 2$ and $z \leq 3 - y = 3 - r \sin\theta$. Hence

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^2 \int_0^{3-\sin\theta} r dz \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^2 (3 - \sin\theta) r dr \, d\theta \\
&= \int_0^{2\pi} (3 - \sin\theta) d\theta \cdot \int_0^2 r \, dr \\
&= 6\pi \cdot 2 = 12\pi
\end{aligned}$$

Problem 15. *A lamina is bounded by $x = 1 - y^2$ and the coordinate axes. Its density is $\rho(x, y) = |y|$. Find the mass and center of mass.*

Solution The mass is given by integrating the density:

$$m = \int_0^1 \int_0^{1-y^2} y \, dx \, dy = \frac{1}{4}$$

The moments are given by

$$\begin{aligned}
M_y &= \int_0^1 \int_0^{1-y^2} xy \, dx \, dy = \frac{1}{12} \\
M_x &= \int_0^1 \int_0^{1-y^2} y^2 \, dx \, dy = \frac{2}{15}
\end{aligned}$$

The center of mass is then $(M_y/m, M_x/m) = (1/3, 8/15)$.

Problem 16. *Find the area of the cone $z^2 = a^2(x^2 + y^2)$ bounded between the planes $z = 1$ and $z = 2$.*

The cone is naturally parameterized as

$$\vec{r}(\theta, z) = \langle \frac{1}{a} z \cos\theta, \frac{1}{a} z \sin\theta, z \rangle$$

with $0 \le \theta \le 2\pi$, $1 \le z \le 2$. Note that

$$\vec{r}_\theta = \frac{z}{a}\langle -\cos\theta, \sin\theta, 0 \rangle$$

$$\vec{r}_z = \langle 0, 0, 1 \rangle$$

Hence, $\vec{r}_\theta \times \vec{r}_z = \langle \frac{z}{a}\sin\theta, \frac{z}{a}\cos\theta, 0 \rangle$, with magnitude $\frac{z}{a}$. Thus, the area is

$$\int_0^{2\pi} \int_1^2 \frac{z}{a}\, dz\, d\theta = \frac{6\pi}{a}$$

Problem 17. *Rewrite*

$$\int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} dz\, dy\, dx$$

in the order $dx\, dy\, dz$.

Solution We have the inequalities

$$-1 \le x \le 1$$

$$x^2 \le y \le 1$$

$$0 \le z \le 1 - y$$

Since y can be as small as 0 (when $x = 0$), we have $0 \le z \le 1$. Rearranging the final inequality gives $y \le 1 - z$ as well. Finally, if we sketch the region determined by $-1 \le x \le 1$ and $x^2 \le y \le 1 - z$, this is the region bounded by a parabola in the xy-plane. Solving for x in the equality case gives $x = \pm\sqrt{y}$, so our integral is

$$\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy\, dz$$

Problem 18. *Evaluate*

$$\iiint xyz dV$$

where E is the region between spheres of radius 2 and 4 around the origin and above the cone $\varphi = \pi/3$.

Solution This is best done in spherical coordinates, in which we have

$$\int_0^{2\pi} \int_0^{\pi/3} \int_2^4 (\rho\cos\theta\sin\varphi)(\rho\sin\theta\sin\varphi)(\rho\cos\varphi)\rho^2 \sin\varphi d\rho\, d\varphi\, d\theta$$

By inspection, the integral in θ is zero, so the overall integral is zero.

Problem 19. *Evaluate*

$$\int_0^1 \int_0^1 e^{\max\{x^2, y^2\}}\, dy\, dx$$

Solution Note $x^2 \ge y^2$ if $x \ge y$, and vice-versa. We integrate over two regions separately:

$$\int_0^1 \int_0^x e^{x^2}\, dy\, dx + \int_0^1 \int_x^1 e^{y^2}\, dy\, dx$$

See problem 13 for the evaluation of these two integrals.

Chapter 16

Problem 20. *Evaluate $\int_C x\, ds$ along the curve $y = x^2$ from $(0, 0)$ to $(1, 1)$.*

Solution The natural parameterization of this curve is $\vec{r}(t) = \langle t, t^2 \rangle$ with $t \in [0,1]$. Hence

$$ds = |\vec{r}'(t)|dt = \sqrt{1+4t^2}dt$$

Thus,

$$\int_C x\,ds = \int_0^1 t\sqrt{1+4t^2} = \frac{5\sqrt{5}-1}{12}$$

after using the substitution $u = 1+4t^2$.

Problem 21. *Find the work done by the vector field* $\vec{F} = \langle z, x, y \rangle$ *along*

(a) the line segment from $(3,0,0)$ *to* $(-3,0,1)$

(b) the helix $x = 3\cos \pi t, y = 3\sin \pi t, z = t, 0 \leq t \leq 1$

Is the field conservative?

Solution

(a) This line segment is parameterized by $\vec{r}(t) = \langle 3,0,0 \rangle(1-t) + \langle -3,0,1 \rangle t = \langle 3-6t, 0, t \rangle$. Hence $d\vec{r} = \langle -6, 0, 1 \rangle dt$ and

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle t, 3-6t, 0 \rangle \cdot \langle -6,0,1 \rangle dt = \int_0^1 -6t = -3$$

(b) Here

$$W = \int_0^1 \langle t, 3\cos \pi t, 3 \sin \pi t \rangle \cdot \langle -3\pi \cos \pi t, 3\pi \cos \pi t, 1 \rangle dt = \int_0^1 -3\pi t \cos \pi t + 9\pi \cos^2 \pi t + 3 \sin \pi t \, dt = \frac{12}{\pi} + \frac{9\pi}{2}$$

(c) The above paths have the same start point and endpoint, but the work done by $\vec{F}$ is not the same. Hence the field is not conservative.

Problem 22. *Find a potential for* $\langle (1+xy)e^{xy}, e^y + x^2 e^{xy} \rangle$.

Solution Integrating the second component in y gives $f(x,y) = e^y + xe^{xy} + k(x)$. Differentiating in x gives

$$(1+xy)e^{xy} = f_x = 0 + xe^{xy}y + e^{xy} + k'(x) \implies k'(x) = 0$$

So any constant k, e.g. 0 works.

Problem 23. *Evaluate the line integral*

$$\int_C \sqrt{1+x^3}dx + 2xydy$$

where C *is the triangle with vertices* $(0,0)$, $(1,0)$ *and* $(1,3)$.

Solution Using Green's theorem with $P = \sqrt{1+x^3}$ and $Q = 2xy$, we can write

$$\int_C \sqrt{1+x^3}dx + 2xydy = \iint_T 2y\,dA$$

where T is the triangle. Putting in bounds, this is

$$\int_0^1 \int_0^{3x} 2y\,dy\,dx = \int_0^1 9x^2\,dx = 3$$

Problem 24. *Is there a vector field $\vec{G}$ such that*

$$\operatorname{curl} \vec{G} = \langle 2x, 3yz, -xz^2 \rangle?$$

Solution No, since the divergence of $\operatorname{curl} \vec{G}$ is $2+3z-2xz \neq 0$, and it's a general fact that the divergence of a curl is zero.

Problem 25. *Verify the conclusion of the divergence theorem for the vector field $\vec{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle$ on the surface S given by $x^2 + y^2 = 1, 0 \leq z \leq 2$.*

Solution Our surface is a cylinder. The divergence is

$$\nabla \cdot F = 3x^2 + 3y^2 + 3z^2$$

and so the integral of the divergence is (in cylindrical coordinates)

$$\iiint_E \operatorname{div} \vec{F} dV = \int_0^{2\pi} \int_0^1 (\int_0^2 3r^2 + 3z^2)\, r\, dz\, dr\, d\theta = 11\pi$$

Now for the surface integral. We have three components:

- The bottom surface has outward normal given by $-\vec{k}$. Note that the field is

  $$\vec{F}(x, y, 0) = \langle ..., ..., 0 \rangle$$

  and so $\vec{F} \cdot \vec{n} = 0$; integrating zero gives zero.
- The top surface has outward normal $\vec{k}$, and the field is

  $$\vec{F}(x, y, 0) = \langle ..., ..., 2^3 \rangle$$

  Hence, the dot product is 8. We then have

  $$\iint_T 8 dS = 8 \operatorname{Area}(T) = 8\pi$$

  where T is the top disk.
- The side is given by the parameterization $\langle \cos\theta, \sin\theta, z \rangle$, and a computation (or just studying this geometrically!) leads us to $\vec{n} = \langle \cos\theta, \sin\theta, 0 \rangle$. Hence we have

  $$\iint_{\text{side}} \vec{F}(x, y, z) \cdot \vec{n} dS = \int_0^{2\pi} \int_0^2 \cos^4\theta + \sin^4\theta\, dz\, d\theta = 3\pi$$

Adding the three contributions gives $0 + 8\pi + 3\pi = 11\pi$, exactly as the divergence theorem says.

Problem 26. *Evaluate the surface integral*

$$\iint_S z dS$$

where S is the surface given by $x^2 + y^2 = z$ below $z = 4$.

Solution The surface is parameterized by $\langle x, y, g(x, y) \rangle$, and so

$$dS = \sqrt{1 + g_x^2 + g_y^2} dA$$

Here $g(x, y) = x^2 + y^2$, so

$$dS = \sqrt{1 + 4x^2 + 4y^2}$$

Our domain is the disk D of radius 2 around the origin, so

$$\iint_S z dS = \iint_D (x^2 + y^2) \sqrt{1 + 4x^2 + 4y^2} dA$$

This is best done in polar coordinates, in which it is

$$\int_0^{2\pi} \int_0^1 r^3 \sqrt{1 + 4r^2} dr\, d\theta = \frac{\pi}{60}(1 + 391\sqrt{17})$$